

# Thermodynamic Volume of Kerr–bolt–AdS Spacetime

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## Abstract

In theories of gravity where the cosmological constant defines a thermodynamic variable, the pressure, it has been shown that solutions of Einstein’s equations have a corresponding thermodynamic volume. In general, the expression for the volume is not the same as the one arising from naive geometrical considerations. Both rotation and nut charge are properties known in the literature to give non-geometric thermodynamic volumes, and so we combine the two in a single example and compute the volume for the Kerr–bolt–AdS spacetime, presenting a new expression that generalizes the previously known non-geometric cases.

# 1 Introduction

Black hole thermodynamics [1, 2, 3, 4] is a subject long known for leading to deep insights into the nature of both classical and quantum gravity. In the last few years the standard story of how one associates gravitational quantities with thermodynamic ones has been extended<sup>1</sup>, by allowing the cosmological constant  $\Lambda$  to be a thermodynamic variable, the pressure,  $p = -\Lambda/8\pi$  (we use units such that  $G = c = \hbar = k_B = 1$ ). In this extended thermodynamics, instead of being the internal energy  $U$ , the mass  $M$ , is identified [9] with the enthalpy:  $M = H \equiv U + pV$ .

Given the association of the pressure with the cosmological constant one may wonder what the conjugate thermodynamic volume corresponds to. It was shown in refs. [9, 10, 11] that for static black holes with horizon radius  $r_h$  the thermodynamic volume  $V$  is the geometric volume<sup>2</sup>  $V_{geo} = \frac{4}{3}\pi r_h^3$ . However, this is not true in general; for example, for the Kerr–AdS black holes the thermodynamic volume and the geometric volume are not the same [11]. More recently, ref. [17] proposed that the extended thermodynamics should hold not just for black hole spacetimes but for *any* spacetime. That is, one should associate the mass with the enthalpy in any system where the cosmological constant is dynamical. As examples, the Taub–NUT–AdS and Taub–bolt–AdS spacetimes were shown to also be cases where the thermodynamic volume is different from the geometric one. The Taub–NUT–AdS spacetime is an extreme example of this, because while the geometric volume vanishes, the thermodynamic volume is finite.

In light of the work in refs. [11, 17], it is interesting to explore more examples in which the thermodynamic volume and the geometric one are not the same. Having seen two examples in the literature where this is the case — Taub–NUT/bolt–AdS and the Kerr–AdS black hole — we consider here a spacetime that is a mixture of the two. We explore the impact of having both rotation (characterized by rotation parameter  $a$ ) and nut charge (characterized by  $n$ ) in order to try and better understand their contribution in the thermodynamic volume. To this end we study the Kerr–bolt–AdS spacetimes in four dimensions<sup>3</sup> [24].

The rest of the paper is organized as follows: we first present the Kerr–bolt–AdS spacetime solutions, which have both rotation and nut charge, along with some of their known thermodynamics. Then using the extended thermodynamics, taking  $p$  as dynamical and associating the mass with the enthalpy, we compute the thermodynamic volume  $V$ . The paper ends with a discussion of our results.

## 2 The Geometry

The metric for the (Euclideanized) Kerr–bolt–AdS spacetime takes the form [24]:

$$ds^2 = \frac{\mathcal{V}(r)(d\tau - (2n \cos \theta - a \sin^2 \theta)d\phi)^2 + \mathcal{H}(\theta) \sin^2 \theta (a d\tau - (r^2 - n^2 - a^2)d\phi)^2}{\chi^4(r^2 - (n + a \cos \theta)^2)} + (r^2 - (n + a \cos \theta)^2) \left( \frac{dr^2}{\mathcal{V}(r)} + \frac{d\theta^2}{\mathcal{H}(\theta)} \right), \quad (1)$$

<sup>1</sup>For a selection of references where this has been explored, see refs. [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. See also the reviews in refs. [19, 20]. For references where this has been explored in the context of conformal gravity and Lovelock gravity, see refs. [21] and [22], respectively.

<sup>2</sup>A definition for the volume of a stationary black hole was proposed in [23], and they obtain this same result for static holes, but for non-static black holes, the expression they find is different.

<sup>3</sup>As we review in the next section, there are no Kerr–NUT–AdS spacetimes.

where the metric functions are given by

$$\mathcal{H}(\theta) = 1 + \frac{qn^2}{l^2} + \frac{(2n + a \cos \theta)^2}{l^2}, \quad (2)$$

$$\mathcal{V}(r) = \frac{r^4}{l^2} + \frac{((q-2)n^2 - a^2 + l^2)r^2}{l^2} - 2mr - \frac{(a^2 - n^2)(qn^2 + l^2 + n^2)}{l^2}. \quad (3)$$

The parameter  $n$  is the nut charge,  $m$  the mass parameter, and  $a$  the rotation parameter. The cosmological constant,  $\Lambda$ , sets the length scale  $l$  via  $\Lambda = -3/l^2$ . The parameters  $q$  and  $\chi$  as well as the periodicity of  $\tau$  are fixed by removing the conical singularities where the  $\tau$  fibre degenerates. Requiring smoothness of the metric in the  $(\theta, \phi)$  section leads to  $q = -4$  and  $\chi = \sqrt{1 + a^2/l^2}$ . Requiring smoothness in the  $(r, \tau)$  section implies that  $\tau$  has period given by  $2\pi/\kappa$  with<sup>4</sup>

$$\kappa = \frac{\mathcal{V}'(r_+)}{2\chi^2 (r_+^2 - r_n^2)}. \quad (4)$$

Here  $r_+$  is the location where the foliation breaks down, i.e.,  $\mathcal{V}(r_+) = 0$  and  $r_n = \sqrt{a^2 + n^2}$  is the location of the nut, where the area of the surfaces orthogonal to the  $(r, \tau)$  section vanishes [24]. The metric reduces to both the Kerr–AdS black hole and Taub–bolt–AdS when  $n = 0$  and  $a = 0$ , respectively<sup>5</sup>.

There is subtlety in fixing the periodicity of  $\tau$  however, because regularity along the Misner string [27] singularity, which runs along the  $z$ -axis from the nut to infinity, requires  $\tau$  to have period  $8\pi n$ . The resolution is to use  $\mathcal{V}(r_+) = 0$  to determine the mass parameter  $m$  in terms of  $r_+$ , and then equate the two periods to give a quartic constraint on  $r_+$  in terms of  $a$ ,  $n$  and  $l$ . The result for  $r_+$  is given in the Appendix, and the expression for  $m$  from the constraint is

$$m = \frac{r_+^2 (-a^2 + l^2 - 6n^2) + (n^2 - a^2) (l^2 - 3n^2) + r_+^4}{2l^2 r_+}. \quad (5)$$

We will take the period of  $\tau$  to be  $2\pi/\kappa$  for computing the temperature. One might be curious what happens when  $r_+ = r_n$ , the analogue of the NUT solution in the non-rotating case. Regularity of the solution demands that  $\mathcal{V}(r)$  have a double root, but this requirement turns out to be incompatible with the periodicity constraints unless  $a = 0$ , thus  $r_+ > r_n$  (there are no regular Kerr–NUT or Kerr–NUT–AdS solutions [24, 28])<sup>6</sup>.

### 3 Gravitational Thermodynamics and the Volume

We are now in a position to do the gravitational thermodynamics. The on-shell (Euclidean) action was found in ref. [24], and the result is

$$I = -\pi \frac{(r_+^2 - n^2 - a^2) [r_+^4 - (a^2 + l^2)r_+^2 + (n^2 - a^2)(3n^2 - l^2)]}{(3r_+^4 + (l^2 - a^2 - 6n^2)r_+^2 + (n^2 - a^2)(3n^2 - l^2)) \chi^2}. \quad (6)$$

Using the thermodynamic relations we can compute the entropy from this expression via  $S = \beta \partial_\beta I - I =$

<sup>4</sup>Note that formula (4) differs from that of ref. [24] by a factor of  $2\pi$  in the denominator and that  $\chi$  in ref. [24] is 1 over what we have here (which matches refs. [25] and [26]).

<sup>5</sup>See the discussion in the next section about matching conventions.

<sup>6</sup>Sometimes in the literature the metric (1) is referred to as Kerr–NUT–AdS, but here we differentiate NUT vs. bolt and so refer to the spacetime as Kerr–bolt–AdS.

$\frac{1}{T}(\overline{M} + \overline{\Omega}J) - I$  where the inverse temperature  $\beta = 1/T = 2\pi/\kappa$ ,  $\overline{M}$  is the energy of the spacetime at infinity (the mass),  $J$  is the conserved charge associated with the Killing vector  $\partial/\partial\phi$ , and  $\overline{\Omega}$  is the angular velocity of the horizon measured in a frame rotating at infinity, given by  $\overline{\Omega} = a/(r_+^2 - a^2 - n^2)$ . Both  $\overline{M}$  and  $J$  were found in ref. [24] to be  $\overline{M} = \frac{m}{\chi^4}$  and  $J = \frac{ma}{\chi^4}$ , and so we can use these plus the expression for  $m$  in (5) to compute the entropy, which is

$$S = \frac{\pi \left( r_+^4 (-4a^2 + l^2 - 15n^2) + (n^2 - a^2)^2 (3n^2 - l^2) + r_+^2 (a^2 + 3n^2)^2 + 3r_+^6 \right)}{\chi^2 (r_+^2 (-a^2 + l^2 - 6n^2) + (n^2 - a^2)(3n^2 - l^2) + 3r_+^4)}. \quad (7)$$

There is a subtlety here pointed out in ref. [29]. If one computes the angular velocity  $\overline{\Omega}$  in a frame that is rotating at infinity then the First Law of Thermodynamics does not hold. Instead, one needs  $\overline{\Omega}$  computed in a frame that is non-rotating at infinity, which is found by taking  $\phi \rightarrow \phi + l^{-2}a\tau$  in (1). This shift changes both  $\overline{\Omega}$  and  $\overline{M}$ , and gives the relationship between the rotating and non-rotating frame quantities to be

$$\overline{\Omega} = \Omega + \frac{a}{l^2}, \quad (8)$$

$$\overline{M} = M + \frac{a}{l^2}J. \quad (9)$$

Now following refs. [9, 11, 17] we take the cosmological constant of the spacetime to supply a thermodynamic pressure,  $p = -\Lambda/8\pi = 3/(8\pi l^2)$ , and label its conjugate volume  $V$ . In this extended thermodynamics the mass  $M$ , instead of being the internal energy  $U$ , is identified with the enthalpy,  $M = H \equiv U + pV$ . The First Law is then

$$dH = TdS + \Omega dJ + Vdp. \quad (10)$$

One could proceed to find the volume using the thermodynamic relation  $V = (\partial H/\partial p)|_{S,J}$ . Instead, we simplify matters by using a Smarr relation [5, 9, 11, 17, 29, 30] which follows entirely from scaling arguments:

$$\frac{H}{2} - TS - \Omega J + pV = 0. \quad (11)$$

We note that to find the transformation of  $V$  between the non-rotating and rotating frame we can use (8) and (9) in (11) to find  $\overline{V} = V + \frac{a}{2l^2p}J$ .

From (11) we find

$$V = \frac{2\pi l^2 (r_+^6 (a^2 - 2l^2) + r_+^4 (-2a^4 + a^2 (l^2 - 7n^2) + 8l^2 n^2))}{3r_+ (a^2 + l^2)^2 (a^2 + n^2 - r_+^2)} + \frac{2\pi l^2 (r_+^2 (a^6 + 10a^4 n^2 + a^2 (-4l^4 + 16l^2 n^2 + 3n^4) - 6l^2 n^4) + a^2 (a^2 - n^2) (l^2 - 3n^2) (a^2 + 4l^2 + n^2))}{3r_+ (a^2 + l^2)^2 (a^2 + n^2 - r_+^2)}, \quad (12)$$

where  $r_+$  is given in the Appendix.

Before proceeding to understand this result, let us check some special cases. Putting  $a = 0$  above we find that

$$V|_{a=0} = \frac{4}{3}\pi (r_b^3 - 3n^2 r_b), \quad (13)$$

where  $r_b = r_+|_{a=0}$  – see the Appendix. This is the volume of Taub-bolt-AdS found in ref. [17]. We note that in the metric (1), we recover the metric used in ref. [17] after changing the sign of the azimuthal coordinate

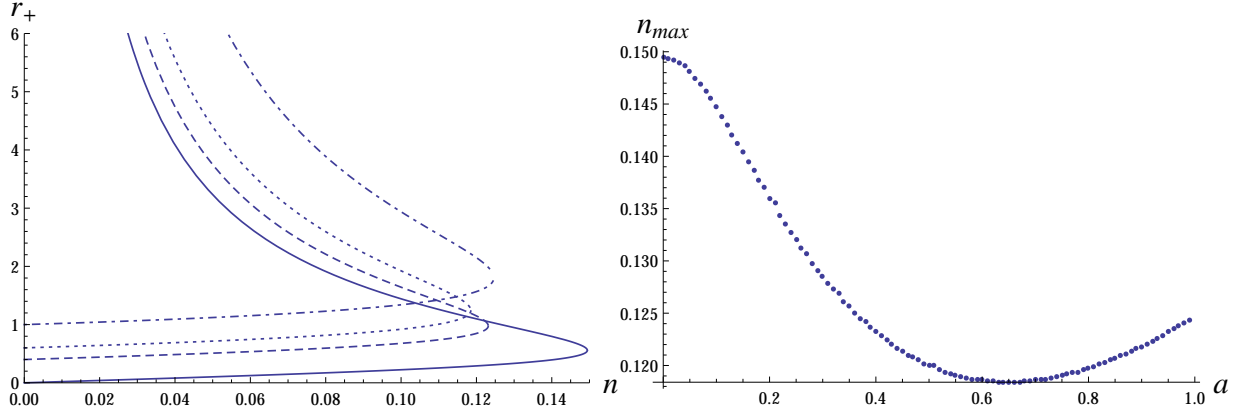


Figure 1:  $r_+$  as a function of  $n$ . The (approximate) maximum value of  $n_{max}$  depends on  $a$  (taking  $l = 1$ ), and we connect the upper and lower branch for  $r_+$  at this point. The curves correspond to: solid  $a = 0$ , dashed  $a = 0.4$ , dotted  $a = 0.6$ , dot-dashed  $a = 1.0$ . Also shown is the value of  $n_{max}$  for various values of  $a$ .

$\phi \rightarrow -\phi$ .

Putting  $n = 0$  we recover the Kerr-AdS black hole metric, thermodynamics<sup>7</sup> and volume as in ref. [11]:

$$V|_{n=0} = \frac{2\pi l^2 (-r_+^4 (a^2 - 2l^2) + a^2 r_+^2 (a^2 + l^2) + a^2 l^2 (a^2 + 4l^2))}{3(a^2 + l^2)^2 r_+}, \quad (14)$$

where  $r_+$  is now dependent on the mass parameter  $m$  (see Appendix), as there is no additional regularity constraint to eliminate  $m$  as an independent parameter. Care must be taken in comparing (14) with the results in ref. [11] due to different conventions. In particular, the  $\tau$  coordinate in (1) is given by  $(1 - \tilde{a}^2 \tilde{l}^{-2}) \tilde{\tau}$  where a tilde denotes their symbols, and also one must take  $\tilde{a} \rightarrow ia$ , which comes from Euclideanisation (and so one must be careful in this when comparing the angular velocities which are to be computed in Lorentzian signature), and  $\tilde{\Lambda} \rightarrow -\Lambda$  which comes from a difference in the sign of the cosmological constant term in the action<sup>8</sup>.

## 4 Exploring the Volume Formula

With (12) in hand we can begin to explore its behavior with respect to the various parameters we have:  $a$ ,  $n$ ,  $r_+$  and  $l$ . First, however, we need to understand the allowed ranges of the parameters. We plot in Figure 1  $r_+$  as a function of  $n$  for various values of  $a$ , setting  $l = 1$ . It is the ratio  $a/l$  that is relevant, but for simplicity we have taken  $l = 1$  and so allow  $0 < a < 1$ .

Note that if one writes the metric (1) in Lorentzian signature (or rather, derives (1) as the Euclidean signature version) then one must take  $a \rightarrow ia$  and so, in particular,  $\chi \rightarrow \sqrt{1 - \alpha^2/l^2}$  and we see that when  $\alpha^2 = l^2$  the metric becomes singular [32], thus the rotation parameter is restricted to  $\alpha^2 < l^2$ . Additionally, the allowed values for  $r_+$  are such that it remains real and greater than  $r_n = \sqrt{a^2 + n^2}$ , meaning that there is a certain maximum value  $n$ , denoted  $n_{max}$ . In Figure 1 we have plotted the approximate value of  $n_{max}$  for various values of  $a$ , found by numerically scanning for the value of  $n$  for which  $\text{Im}(r_+(n))$  first becomes

<sup>7</sup>We point out that the gravitational and thermodynamic stability of Kerr-AdS black holes in five and higher dimensions was studied in [31] but with no pressure term.

<sup>8</sup>In ref. [11] they choose to work with thermodynamic variables  $\Lambda$  and  $\Theta$ , instead of  $p$  and  $V$ . When matching their formulae, one must also take  $V \rightarrow +8\pi\Theta$

non-zero. To get the full behavior of  $r_+(n)$  we must recall that  $r_+$  in general satisfies a quartic constraint and we take the two largest roots, where the cross-over between the two branches occurs at  $n_{max}$ . We note how  $n_{max}$  decreases as a function of  $a$  for  $a < 0.65$ , but increases as a function of  $a$  for  $a > 0.65$ .

Using this we turn to plotting the volume as a function of  $n$  for various values of  $a$ , and taking  $l = 1$  — the result is in Figure 2 (a). We see that the volume appears to be a smooth function of  $n$  and positive-definite. The turning point of  $V$  corresponds to when  $n$  has reached  $n_{max}$  for that particular value of  $a$ . We can also look at the behavior of  $V$  as a function of  $l$ . In the case when  $n = 0$  we have plotted the result in Figure 2 (b). We find again that the volume appears to be positive-definite. It is difficult to show analytically that the volume is always positive-definite, but our plots strongly indicate this to be true<sup>9</sup>.

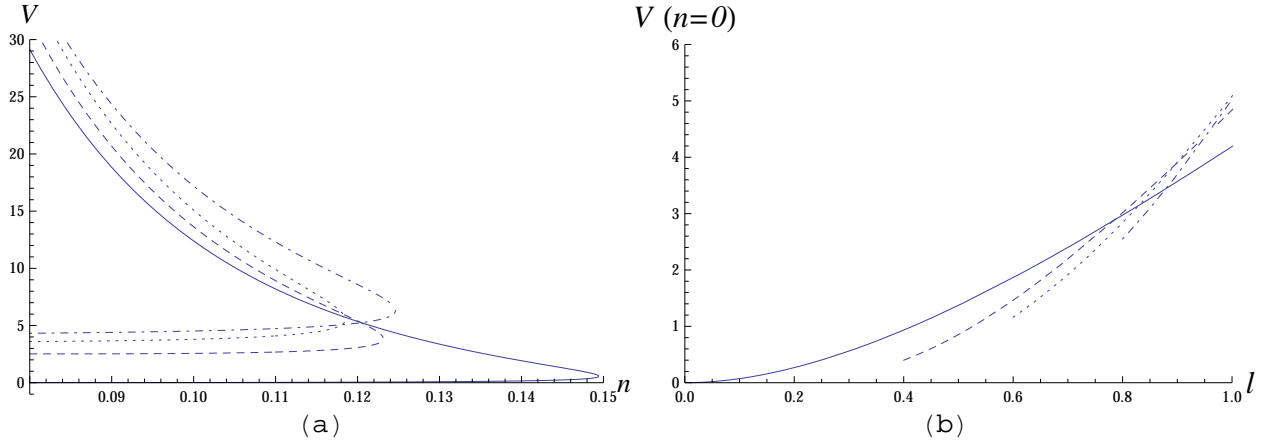


Figure 2: (a)  $V$  as a function of  $n$ . The curves correspond to: solid  $a = 0$ , dashed  $a = 0.4$ , dotted  $a = 0.6$ , dot-dashed  $a = 1.0$ . (b)  $V$  as a function of  $l$  when  $n = 0$ . Here  $r_+$  depends on  $m$  which is an independent variable. We have taken  $m = 1$ . The curves correspond to: solid  $a = 0$ , dashed  $a = 0.4$ , dotted  $a = 0.6$ , dot-dashed  $a = 0.8$ .

One observation is that the thermodynamic volume is not just the sum of the volumes for the Kerr–AdS and Taub–bolt spacetime

$$V \neq V|_{a=0} + V|_{n=0}. \quad (15)$$

We can also expand the volume about both small  $a$  and small  $n$ . We consider the former first. We make a series expansion for  $m$  about  $a = 0$  and use the condition  $\mathcal{V}(r_+) = 0$  to determine the coefficients; to second order in  $a$  we find

$$\begin{aligned} m &= \sum_{i=0}^{\infty} f_i a^i = f_0 + f_1 a + f_2 a^2 + \mathcal{O}(a^3), \\ f_0 &= \frac{l^2 n^2 - 3n^4 + l^2 r_+^2 - 6n^2 r_+^2 + r_+^4}{2l^2 r_+}, \quad f_1 = 0, \\ f_2 &= \frac{3n^2 - l^2 - r_+^2}{2l^2 r_+}. \end{aligned} \quad (16)$$

Using the condition that  $8\pi n = 2\pi/\kappa$  (with  $\kappa$  given in (4)) we can find a series expansion for  $r_+$  about  $a = 0$ .

<sup>9</sup>Positive-definiteness is not guaranteed for the thermodynamic volume of a spacetime. For an example of a negative volume, see ref. [17] and the interpretation therein. Also, in ref. [33], a negative thermodynamic volume for a black hole in Gauss-Bonnet gravity was found.

It becomes rather messy at second order so we show it here only to first order:

$$r_+ = \sum_{i=0}^{\infty} g_i a^i = g_0 + g_1 a + \mathcal{O}(a^2), \quad (17)$$

$$g_0 = r_b \equiv \frac{l^2}{12n} \left( 1 \pm \sqrt{1 - 48 \frac{n^2}{l^2} + 144 \frac{n^4}{l^4}} \right), \quad g_1 = 0.$$

where  $r_b = r_+|_{n=0}$  is just the solution to the quadratic equation for the Taub–bolt–AdS spacetime [17]. With (16) and (17) we can expand the volume to first order in  $a$  and find

$$\begin{aligned} V &\approx \frac{4}{3} \pi (g_0^3 - 3n^2 g_0) + 4\pi (g_0^2 - n^2) g_1 a + \mathcal{O}(a^2) \\ &= \frac{4}{3} \pi (r_b^3 - 3n^2 r_b) + \mathcal{O}(a^2). \end{aligned} \quad (18)$$

which tells us that the volume is just that of Taub–bolt–AdS [17] to first order in  $a$ .

We may repeat the above procedure for small  $n$ , which turns out to be more interesting. We find from  $\mathcal{V}(r_+) = 0$  that

$$\begin{aligned} m &= \sum_{i=0}^{\infty} h_i n^i = h_0 + h_1 n + h_2 n^2 + \mathcal{O}(n^3), \\ h_0 &= \frac{-a^2 l^2 - a^2 r_+^2 + l^2 r_+^2 + r_+^4}{2l^2 r_+}, \quad h_1 = 0, \\ h_2 &= \frac{3a^2 + l^2 - 6r_+^2}{2l^2 r_+}. \end{aligned} \quad (19)$$

Using the condition  $8\pi n = 2\pi/\kappa$  and writing

$$r_+ = \sum_{i=0}^{\infty} k_i n^i, \quad (20)$$

we find that either  $k_0 = \pm a$  or  $k_0 = 0$ . The latter is inconsistent, so we have (taking the positive sign)

$$k_0 = a, \quad k_1 = 2, \quad k_2 = \frac{29a^2 - 3l^2}{2a(a^2 + l^2)}. \quad (21)$$

From (19) and (20) we find  $V$  to first order in small  $n$  to be

$$V \approx \frac{8\pi a l^4}{3(a^2 + l^2)} - \frac{4\pi (2a^4 l^2 - 5a^2 l^4 + 3l^6)}{3(a^2 + l^2)^2} n + \mathcal{O}(n^2). \quad (22)$$

We note that (22) at 0th order is not (14), the result for the Kerr–AdS black hole [11]. This is because small  $n$  and  $n = 0$  are not the same. For small  $n$  we still have to impose the regularity condition due to the Misner string, and so we have the  $8\pi n = 2\pi/\kappa$  constraint which allows us write  $r_+$  in terms of  $a$ ,  $l$  and small  $n$ , obtaining (22) for the volume. When  $n = 0$  there is no Misner string condition and so no additional constraint on  $r_+$ . We write  $V$  in terms of  $m$  when  $n = 0$  and then substitute  $m$  in terms of  $r_+$ ,  $a$  and  $l$  (which is the 0th order term in (19)) to obtain (14). We also note that there is a first order contribution to the volume for small  $n$  in contrast to the small  $a$  expansion (18), in which  $a$  doesn't appear at first order.

## 5 Conclusion

We have computed the thermodynamic volume for a class of Kerr–bolt–AdS spacetimes and explored its behavior as a function of the rotation parameter  $a$  and the nut charge  $n$ . Our plots in Figure 2 strongly suggest that this volume is positive–definite, as expected. Additionally we found that the thermodynamic volumes for the two limiting cases ( $a = 0$  and  $n = 0$ ) do not simply add to give  $V$ . Taking the small  $a$  expansion we found that  $a$  enters the volume at second order, in contrast to the small  $n$  expansion in which there is a first order contribution from  $n$ .

There are many future directions of research one can explore. One possibility would be to extend our analysis to higher dimensions, though care has to be taken when dealing with even vs. odd dimension. It is interesting to continue to study other examples where the thermodynamic and the geometric volume differ, and perhaps eventually classify the various characteristic that go into making the thermodynamic volume. There is also the matter of understanding and interpreting the thermodynamic volume in the context of gauge/gravity duality, so one could consider our thermodynamics in the Kerr/CFT correspondence [26, 34] to see what could be learned. We leave these matters for future investigation.

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## Appendix

For completeness we give the full expression for  $r_+$ , which is constrained by the regularity condition explained in Section 2 to be a solution of the quartic polynomial defined by  $\beta = 1/T = 8\pi n = \kappa/2\pi$ , with  $\kappa$  given in (4). We can write this constraint in the standard form

$$Ar_+^4 + Br_+^3 + Cr_+^2 + Dr_+ + E = 0,$$

for which known formulae exist for the solution, and the roots may be written

$$\begin{aligned} r_{+1,2} &= -\frac{B}{4A} + Z \pm \frac{1}{2}\sqrt{-4Z^2 - 2W - \frac{Y}{Z}}, \\ r_{+3,4} &= -\frac{B}{4A} - Z \pm \frac{1}{2}\sqrt{-4Z^2 - 2W + \frac{Y}{Z}}, \end{aligned}$$



where we take  $r_{+1,2}$  as the two branches of  $r_+$  in the paper and

$$\begin{aligned} W &= \frac{8AC - 3B^2}{8A^2}, & Y &= \frac{B^3 - 4ABC + 8A^2D}{8A^3}, \\ Z &= \frac{1}{2}\sqrt{-\frac{2}{3}W + \frac{1}{3A}\left(Q + \frac{\Delta_0}{Q}\right)}, & Q &= \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}, \\ \Delta_0 &= C^2 - 3BD + 12AE, \\ \Delta_1 &= 2C^3 - 9BCD + 27B^2E + 27AD^2 - 72ACE. \end{aligned}$$

For Kerr–bolt–AdS spacetime the coefficients of the quartic for  $r_+$  are

$$\begin{aligned} A &= 6n, & B &= -(a^2 + l^2), \\ C &= 2(l^2 - a^2 - 6n^2), & D &= (a^2 + l^2)(a^2 + n^2), \\ E &= 2n(l^2 - 3n^2)(a^2 - n^2). \end{aligned}$$

Taking  $a = 0$  in the metric reduces the constraint on  $r_+$  to be quadratic and we find [17]

$$r_b = r_+|_{a=0} = \frac{l^2}{12n} \left( 1 \pm \sqrt{1 - 48\frac{n^2}{l^2} + 144\frac{n^4}{l^4}} \right).$$

Taking  $n = 0$  in the metric we can no longer impose the condition on  $m$ , so it must now be taken as an independent parameter and we may write the quartic constraint on  $r_+|_{n=0}$  in the standard form above with

$$\begin{aligned} A &= \frac{1}{l^4}, & B &= 0, \\ C &= 1 - \frac{a^2}{l^2}, & D &= -2m, \\ E &= -a^2. \end{aligned}$$

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